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FROM QUANTUM AFFINE GROUPS TO THE EXACT DYNAMICAL CORRELATION FUNCTION OF THE HEISENBERG MODEL

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The exact form factors of the Heisenberg models XXX and XXZ have been recently computed through the quantum affine symmetry of XXZ model in the thermodynamic limit. We use them to derive an exact formula for the contribution of two spinons to the dynamical correlation function of XXX model at zero temperature.

1. Introduction

This letter is a short summary of our results on the two-spinon dynamical correlation function (DCF) of the Heisenberg model[1]. Scattering cross sections of neutrons with spin chains are directly proportional to dynamical correlation functions. The latter, in turn, are expressed as series in terms of form factors of local spin operators. Unfortunately, unlike those in quantum field theories,[2] form factors of spin chains are very complicated because of the non-relativistic dispersion relations of these lattice models. For more details on the usefulness of form factors and DCF see Refs.[3,4,5,6,7,8,9]. In a recent development however, it has been realized that in the thermodynamic limit the Heisenberg model XXZ becomes symmetric under the quantum affine algebra $U_q(\widehat{sl(2)})$ in the anti-ferromagnetic regime.[10] This puts its resolution on the same footing as that of conformal field theory models. In fact, the bosonization technic of conformal field theory extends to this case and has been

used to compute exact static correlation functions and form factors of spin local operators of XXZ, and also those of XXX after taking the isotropic limit $q \to -1$ of the former.[10] Unfortunately, the latter physical quantities have very cumbersome multi-integral form which limit their usefulness for the exact computation of DCF. However, our main point here is that there is one exception to this latter statement and that is the form factors needed for the contribution of two spinons to the DCF of the Heisenberg model XXX have more tractable form. Therefore we use them to compute this more interesting quantity of the exact two-spinon DCF of XXX in the thermodynamic limit and at zero temperature.

2. Quantum affine symmetry of XXZ model

In this section, we briefly review the XXZ model in the thermodynamic limit and its quantum affine symmetry. This model is defined through its Hamiltonian

$$H_{XXZ} = -\frac{1}{2} \sum_{n=-\infty}^{\infty} (\sigma_n^x \sigma_{n+1}^x + \sigma_n^y \sigma_{n+1}^y + \Delta \sigma_n^z \sigma_{n+1}^z), \tag{1}$$

where $\Delta = (q + q^{-1})/2$ is the anisotropy parameter. Here $\sigma_n^{x,y,z}$ are the usual Pauli matrices acting at the nth position of the formal infinite tensor product

$$W = \cdots V \otimes V \otimes V \cdots, \tag{2}$$

where V is the two-dimensional representation of $U_q(sl(2))$ quantum group. We consider the model in the anti-ferromagnetic regime $\Delta < -1$, i.e., $-1 < q < 0$. The action of H_{XXZ} on W is not well defined due to the appearance of divergences. However, since this model is symmetric under the quantum affine group $U_q(\widehat{sl(2)})$, the eigenspace on which this action becomes well defined is identified with the following level 0 $U_q(\widehat{sl(2)})$ module:

$$\mathcal{F} = \sum_{i,j} V(\Lambda_i) \otimes V(\Lambda_j)^*, \tag{3}$$

where Λ_i and $V(\Lambda_i); i = 0, 1$ are level 1 $U_q(\widehat{sl(2)})$-highest weights and $U_q(\widehat{sl(2)})$-highest weight modules, respectively. Roughly speaking, $V(\Lambda_i)$ is identified with the subspace of the formal semi-infinite space

$$X = \cdots V \otimes V \otimes V, \tag{4}$$

consisting of all linear combinations of spin configurations with fixed boundary conditions such that the eigenvalues of σ_n^z are $(-1)^{i+n}$ in the limit $n \to -\infty$. The eigenspace $\mathcal{F}$ consists of spinon particles[11] $\{|\xi_1, \cdots \xi_n >_{\epsilon_1, \cdots \epsilon_n; i}, n \geq 0\}$. Here i fixes the boundary condition, ξ_j are the spectral parameters, and $\epsilon_j = \pm 1$ are the spins of the spinons. The completeness relation reads:[10]

$$\mathbf{I} = \sum_{i=0,1} \sum_{n \geq 0} \sum_{\epsilon_1, \cdots, \epsilon_n = \pm 1} \frac{1}{n!} \oint \frac{d\xi_1}{2\pi i \xi_1} \cdots \frac{d\xi_n}{2\pi i \xi_n} |\xi_n, \cdots, \xi_1 >_{\epsilon_n, \cdots, \epsilon_1; i} \ _{i; \epsilon_1, \cdots, \epsilon_n} < \xi_1, \cdots, \xi_n|. \tag{5}$$

The actions of H_{XXZ} and the translation operator T, which shifts the spin chain by one site, on $\mathcal{F}$ are given by

$$
\begin{aligned}
T|\xi_1, \cdots, \xi_n >_i &= \prod_{i=1}^{n} \tau(\xi_i)^{-1}|\xi_1, \cdots, \xi_n >_{1-i}, \quad T|0 >_i = |0 >_{1-i}, \\
H_{XXZ}|\xi_1, \cdots, \xi_n >_i &= \sum_{i=1}^{n} e(\xi_i)|\xi_1, \cdots, \xi_n >_i,
\end{aligned}
\tag{6}
$$

where

$$
\begin{aligned}
\tau(\xi) &= \xi^{-1} \frac{\theta_{q^4}(q\xi^2)}{\theta_{q^4}(q\xi^{-2})} = e^{-ip(\alpha)}, \quad p(\alpha) = am(\tfrac{2K}{\pi}\alpha) - \pi/2, \\
e(\xi) &= \frac{1-q^2}{2q}\xi \frac{d}{d\xi} \log \tau(\xi) = \frac{2K}{\pi} \sinh(\frac{\pi K'}{K})dn(\frac{2K}{\pi}\alpha).
\end{aligned}
\tag{7}
$$

Here, $e(\xi)$ and $p(\alpha)$ are the energy and the momentum of the spinon respectively, $am(x)$ and $dn(x)$ are the usual elliptic amplitude and delta functions, with the complete elliptic integrals K and K', and

$$
\begin{aligned}
q &= -\exp(-\pi K'/K), \\
\xi &= ie^{i\alpha}, \\
\theta_x(y) &= (x;x)_\infty (y;x)_\infty (xy^{-1};x)_\infty, \\
(y;x)_\infty &= \prod_{n=0}^{\infty}(1 - yx^n).
\end{aligned}
\tag{8}
$$

This means, $\sigma^{x,y,z}(t,n)$ at time t and position n are related to $\sigma^{x,y,z}(0,0)$ at time 0 and position 0 through:

$$
\sigma^{x,y,z}(t,n) = \exp(itH_{XXZ})T^{-n}\sigma^{x,y,z}(0,0)T^n \exp(-itH_{XXZ}).
\tag{9}
$$

3. Two-spinon dynamical correlation function of XXX model

Here we first define one of the components of the DCF in the case of the XXZ model , where the spinon picture is well understood, and then take its isotropic limit to get *all* non-vanishing components of the DCF of XXX model using its isotropy. We consider the component

$$
S^{i,+-}(w,k) = \int_{-\infty}^{\infty} dt \sum_{n \in Z} e^{i(wt+kn)}{}_i < 0|\sigma^+(t,n)\sigma^-(0,0)|0 >_i,
\tag{10}
$$

where w and k are the neutron energy and momentum transfer respectively, and i corresponds to the boundary condition. Later, we find that the DCF is in fact independent of i. Using the completeness relation, the two-spinon contribution is given by

$$
\begin{aligned}
S_2^{i,+-}(w,k) &= \pi \sum_{n \in Z} \sum_{\epsilon_1,\epsilon_2} \oint \frac{d\xi_1}{2\pi i \xi_1} \frac{d\xi_2}{2\pi i \xi_2} \exp\left(in(k + p(\xi_1) + p(\xi_2))\right) \\
&\times \delta(w - e(\xi_1) - e(\xi_2))_{i+n} < 0|\sigma^+(0,0)|\xi_2, \xi_1 >_{\epsilon_2,\epsilon_1;i+n} {}_{i;\epsilon_1,\epsilon_2} < \xi_1, \xi_2|\sigma^-(0,0)|0 >_i .
\end{aligned}
\tag{11}
$$

It can be put in the following tractable form:

$$
\begin{aligned}
S_2^{i,+-}(w,k) &= \pi \sum_{\epsilon_1,\epsilon_2} \oint \frac{d\xi_1}{2\pi i \xi_1} \frac{d\xi_2}{2\pi i \xi_2} \sum_{n\in Z} \exp\left(2in(k+p(\xi_1)+p(\xi_2))\right) \\
&\times \delta(w-e(\xi_1)-e(\xi_2))\Big(\,_i<0|\sigma^+(0,0)|\xi_2,\xi_1>_{\epsilon_2,\epsilon_1;i}\,_{i;\epsilon_1,\epsilon_2}<\xi_1,\xi_2|\sigma^-(0,0)|0>_i \\
&+ \exp\left(i(k+p(\xi_1)+p(\xi_2))\right)\,_{1-i}<0|\sigma^+(0,0)|\xi_2,\xi_1>_{\epsilon_2,\epsilon_1;1-i} \\
&\quad _{i;\epsilon_1,\epsilon_2}<\xi_1,\xi_2|\sigma^-(0,0)|0>_i\Big).
\end{aligned}
$$

(12)

Now the various form factors invloved in this expression have been computed in Ref. 10. For our purposes, we give only their isotropic limits which are obtained by first making the redefinitions

$$
\begin{aligned}
\xi &= ie^{\frac{\epsilon\beta}{i\pi}}, \\
q &= -e^{-\epsilon}, \quad \epsilon\to 0^+,
\end{aligned}
$$

(13)

with β being the appropriate spectral parameter for XXX model, and then taking the limit $q\to -1$. Therefore one finds:[10]

$$
\begin{aligned}
|\,_i<0|\sigma^+(0,0)|\xi_2,\xi_1>_{--;i}\,|^2\frac{d\xi_1}{2\pi i\xi_1}\frac{d\xi_2}{2\pi i\xi_2} &\to \frac{\Gamma(3/4)^2|A_-(\beta_1-\beta_2)|^2 d\beta_1 d\beta_2}{16\Gamma(1/4)^2|A_+(i\pi/2)|^2|A_-(i\pi/2)|^2\cosh(\beta_1)\cosh(\beta_2)}, \\
p(\xi)\to p(\beta), \quad \text{s.t.} \quad &\cot(p(\beta)) = \sinh(\beta), \quad -\pi\le p(\beta)\le 0, \\
e(\xi)\to e(\beta) = \frac{\pi}{\cosh(\beta)} &= -\pi\sin(p(\beta)),
\end{aligned}
$$

(14)

where

$$
|A_\pm(\beta)|^2 = \exp\left(-\int_0^\infty dx\frac{(\cosh(2x(1-\frac{\delta}{\pi}))\cos(\frac{2x\gamma}{\pi})-1)\exp(\mp x)}{x\sinh(2x)\cosh(x)}\right).
$$

(15)

Here $\Gamma(x)$ is the usual gamma function and $\beta = \gamma + i\delta$, with γ and δ being real. Restricting to the first Brillouin zone, integrating the continuous and discrete delta functions and keeping track of the Jacobian factors, we find that the latter expression is independent of the boundary conditions i (which is henceforth omitted). Moreover, it substantially simplifies to:

$$
S_2^{+-}(w,k) = \frac{\pi^2\Gamma(3/4)^2\Theta(2\pi\sin(k/2)-w)\Theta(w-\pi|\sin(k)|)}{4\Gamma(1/4)^2|A_-(i\pi/2)|^2|A_+(i\pi/2)|^2}\frac{|A_-(\bar\beta_1-\bar\beta_2)|^2}{\sqrt{(2\pi\sin(k/2))^2-w^2}},
$$

(16)

where Θ is the Heaviside step function, and for fixed w and k, $\bar\beta_1$ and $\bar\beta_2$ are the solutions to:

$$
\begin{aligned}
w &= e(\bar\beta_1)+e(\bar\beta_2), \\
k &= -p(\bar\beta_1)-p(\bar\beta_2).
\end{aligned}
$$

(17)

Note that the pair $(\bar\beta_1,\bar\beta_2)$ is identified with the pair $(\bar\beta_2,\bar\beta_1)$. From the isotropy of the Heisenberg model and the inclusion of both sectors $i=0$ and $i=1$, we obtain all non-vanishing components of its DCF from $S_2^{+-}(w,k)$ as:

$$
S_2^{xx}(w,k) = S_2^{yy}(w,k) = S_2^{zz}(k,w) = 4S_2^{+-}(w,k),
$$

(18)

with

$$\sigma^{\pm} = \frac{\sigma^x \pm i\sigma^y}{2}. \tag{19}$$

Despite its square root singularity, $S_2^{+-}(w,k)$ actually vanishes in the vicinity of the upper boundary $w_u = 2\pi \sin(k/2)$ in the dispersion relation of two spinons. Moreover, it diverges in the vicinity of the lower boundary which is given by the des Cloizeaux-Pearson dispersion relation $w_l = \pi|\sin k|$. This behaviour reproduces very well the one obtained previously through the ansatz made in Ref. 5, although now the upper cutoff appears naturally. It would be interesting to investigate to which extent the two-spinon DCF verifies the sum rules which are valid for the full DCF.[12] The extension of this work to the Heisenberg model with higher spin is certainly desirable. In this case, the form factors can in principle be computed through the bosonization of the vertex operators which is now available in Ref. 13.

Acknowledgements

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